LMIs, Interior Point Methods, Complexity Theory, and Robustness Analysis

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Abstract

Let δ_Σ be a measure of the relative stability of a stable dynamical system Σ. Let $\tau_{\mathcal{A}(\Sigma)}$ be **a** measure of the computational efficiency of a particular algorithm A which *verifies* the stability property of X. For two representative cases of Σ, we demonstrate the existence of a particular measure δ_Σ and an algorithm $\mathcal{A}$ such that,

$$\delta_\Sigma \tau_{\mathcal{A}(\Sigma)} = c$$

where c depends possibly on the dimension of the system Σ and parameters which are specific to the algorithm $\mathcal{A}$, but independent of any other system characteristics. In particular, given X and d, one can estimate δ_Σ by *measuring* $\tau_{\mathcal{A}(\Sigma)}$.

1 Introduction

1.1

The fields of control and system theory 011 one hand, and computational complexity on the other, are not generally considered by the researchers of either field to be based on similar principles. Recently, some control and system theorists have begun a serious study of control problems from the computational complexity point of view, e.g., classifying control problems in terms of the complexity class which they belong to [3], [9], [1 I], **[]5], [16].** This line of research is concerned with determining whether a control problem is for example NP-hard, etc. Such results convey the idea that the corresponding system problem, whether it is analysis or synthesis, is computationally *difficult.* One major issue which we believe has not been consider-ed in this direction is the role that the theoretical studies of the computational efficiency of algorithms can play in analyzing systems problems that *can be solved efficiently.* Given a control or system problem that we can solve by means of an algorithm in a reasonable time (for example in time which is proportional to a polynomial of the dimension of the system), what does the *running time* of the algorithm disclose about some of the characteristics of the system under study? In this avenue, suppose that one wants to examine the stability properties of a certain dynamical system and an algorithm is used for this purpose. Thus we use an algorithm which accepts as the input, a description of the system (e. g., in terms of matrices), and produces as an output "yes" or "no, " indicating respectively, whether the system is stable or unstable. Suppose furthermore that the time required for the termination of this algorithm is proportional to the dimension of the system *and* another parameter, denoted by ξ. We would like to show that for certain problems in systems and control theory, there exist algorithms for which the corresponding ξ can be viewed as a certain measure of *robustness, e.g.,* stability margin.

1.2

The results connecting robustness properties of dynamical systems and computational efficiency of algorithms have very interesting implications in system and control theory. It turns out however that these results are a particular way of interpreting the related studies in the complexity theory in terms of condition measures as reported in [1 **3].** The connection can in fact be established at a much deeper level, in a sense that they suggest a unifying framework for studying computational complexity and robustness.

The organization of the paper is as follows. in the next section we initially consider the Lyapunov equation, and demonstrate that the product of the running time of a particular interior point method (ipm) and a robustness measure for linear systems is a constant which depends only on the dimension of the system under study. We then turn our attention to the more genera] problem of checking the positive realness of a transfer matrix. It is shown that the efficiency of the ipm for determining the positive realness of a transfer matrix is again related to certain notions of robustness. The paper is concluded with a brief after-tllought on the implications of the results. The results reported in the paper are based on a section of the manuscript [8].

A few words on the notation. The notation herm(A) denotes the hermitian part of the matrix A, i.e., $\frac{A+A'}{2}$; for two symmetric matrices A and B, $A > B$ ($A \geq B$) indicates that $A - B$ is positive definite (positive semi-

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definite, respectively). As noted previously, $\tau_{A(\Sigma)}$ designates the running time of the algorithm d which verifies the stability properties of the dynamical system Σ. Finally, $f(n) = O(g(n))$ ($f(n) = \Omega(g(n))$) indicates that there exist positive constants c and m such that $0 \leq f(n) \leq cg(n)$ ($0 \leq cg(n) \leq f(n)$), for all $n \geq m$, i.e., $f(n)$ grows slower (faster) than $g(n)$ as n increases.

2 Interior Point Methods and Stability Analysis

2.1

We now begin our study of the relationship) between the computational efficiency of the interior point methods and the stability properties of dynamical systems. We initially present this relationship in terms of the stability criterion for linear time invariant systems expressed in terms of the Lyapunov equation. This is done mainly to establish the conceptual vein which is pursued in the paper; we do not suggest that Lyapunov equations should be solved via the interior point methods (although they certainly can be). Nevertheless, the insight that one obtains from such an analysis, hopefully, justifies the presentation of these results. We subsequently continue exploring the relationship between efficiency/stability in the framework of the Positive Real Lemma. The latter presentation runs parallel to the former case of the Lyapunov equation; the conceptual implications follow as well, although at a deeper level which we shall elaborate on.

2.2

As any student of control theory knows, in order to establish the stability of the origin for the system Σ_1, defined by the matrix $A \in R^{n\times n}$,

$$\Sigma_1: \quad \dot{x} = Ax \tag{2.1}$$

one can check the feasibility of the following system of linear matrix inequalities (LMIs):

$$\mathcal{L}_1: \quad A'P + PA < 0 \tag{2.2}$$
$$P > 0 \tag{2.3}$$

Let us for the moment forget that these matrix inequalities can somehow be solved via a system of linear equations. We approach the problem of finding a feasible point of the set defined by (2.2)- (2.3) via the interior point methods. This will provide us with an opportunity to, rather informally, review the interior point methods (ipms), as well as presenting the idea which shall be generalized in the subsequent section. The material on the ipms which follow have been presented in a more general setting in [10] and [13]; our presentation follows the latter reference.

In order to find a feasible point of (2.2)- (2.3), one can consider instead the following optimization problem,

$$\mathcal{L}_2: \quad \inf \ t \tag{2,4}$$
$$\text{St.} \quad A'P + PA < t(A'\bar{P} + \bar{P}A - I) \tag{2.5}$$
$$P > 0 \tag{2.6}$$
$$t \geq 0 \tag{2.7}$$

where the matrix $\bar{P}$ is chosen to be positive definite, e.g., $\bar{P} = I$. Note that the feasible set of $\mathcal{L}_2$ is a subset of $SR^{n\times n} \times R$. One might wonder why $\bar{P}$ is introduced in (2.5). The reason is that in doing so, a feasible point of $\mathcal{L}_2$ is readily available: $(P_0, t_0) = (\bar{P}, 1)$. Our task is now to initiate the algorithm from $(\bar{P}, 1)$ with the objective value of 1, and try to somehow reduce the objective value to zero (which would be the case if and only if Σ_1 is stable), *without* leaving the feasible region of $\mathcal{L}_2$. This is exactly what an interior point method does (more specifically, we have in mind the barrier method).

Few comments, and a reformulation of $\mathcal{L}_2$ precedes our description of the barrier method. Suppose that we were to solve the following optimization problem:

$$\mathcal{L}_3: \quad \inf \ t \tag{2.8}$$
$$\text{s, t.} \quad A'P + PA < tI \tag{2.9}$$
$$P > 0 \tag{2.10}$$
$$\|P\| < 1 \tag{2.11}$$
$$-1 < t < 2 \tag{2.12}$$

Let $t_{\inf}$ and $t_{\sup}$ denote the value of the infimum and the supremum of the objective functional on the respective region (e. g., $t_{\inf} = 0$ in $\mathcal{L}_2$ if Σ_1 is stable). The value of $t_{\inf}$ in $\mathcal{L}_3$ clearly is a measure of relative stability; intuitively, the more negative one can choose i, the more "stable" Σ_1 is. The lower bound for t and the norm constraint on P are chosen for normalization purposes; otherwise the problem would be unbounded, if feasible. The choice of the upper bound for t would be justifies shortly.

It is not clear however how to "pick" a feasible point for $\mathcal{L}_3$ to initiate the algorithm from. We thus consider instead a combination of $\mathcal{L}_2$ and $\mathcal{L}_3$:

$$\mathcal{L}: \quad \inf \ t \tag{2.13}$$
$$\text{s. t.} \quad A'P + PA < t(A'\bar{P} + \bar{P}A - I) \tag{2.14}$$
$$P > 0 \tag{2.15}$$
$$\|P\| < 1 \tag{2.16}$$
$$-1 < t < 2 \tag{2.17}$$

with $\bar{P} > 0$ and $\|\bar{P}\| < 1$. The initial point $(\bar{P}, 1)$ *is* now readily available as an initial point. Again, the value of $t_{\inf}$ for $\mathcal{L}$ somehow conveys information regarding the relative stability of Σ_1, an observation which shall be made more precise shortly.

Let us denote the feasible region of $\mathcal{L}$ by $\mathcal{F}_L$. Note that $\mathcal{F}_L \subseteq SR_+^{n\times n}$ x R and that it is an open and convex set. It turns out that associated with the set $\mathcal{F}_L$, there is a functional b : interior $\mathcal{F}_L \to R$, which acts as a "self-concordant barrier." [1] 'he term "self concordant" refers to certain properties of the gradient and the Hessian of the functional b evaluated at points in $\mathcal{F}_L$; for the purpose of our presentation, wc shall bypass the exact definition and refer the interested reader to the references given above for the ipm theory.

There arc two important points however that need to be mentioned regarding the functional b. First, if $\{x_k\}_{k>1} \in \mathcal{F}_L$ is a sequence that approaches the boundary of $\mathcal{F}_L$, $b(x_k) \to \infty$ as $k \to \infty$. Second, there is a parameter K associated with b which determines the computational efficiency of the interior point method for minimizing (or maximizing) a linear functional over $\mathcal{F}_L$, the so-called self-collcordant parameter. For brevity, wc shall simply write down the self-concordant barrier for $\mathcal{F}_L$ and its associated self-concordant parameter K. Subsequently, we provide a description of the algorithm for solving $\mathcal{L}$ using b, and its efficiency in terms of the self-collcorda~lt parameter of b.

Let $b : \mathcal{F}_L \to R$ be defined as:

$$b(P,t) = -12\log\det P$$
$$-12\log\det(-A'P - PA + t(A'\bar{P} + \bar{P}A + I))$$
$$-12\log(1 - \|P\|^2) - 12\log(t+1) - 12\log(2-t)$$

Note that indeed when $(P_k, t_k) \to$ boundary of $\mathcal{F}_L$ $b(P_k, t_k) \to \infty$. The associated self-co~lcordallt parameter for the functional b turns out to be,

$$K := \sqrt{12n + 12n + 48} = O(\sqrt{n}) \qquad (2.18)$$

Wc arc now ready to describe the interior point method for solving $\mathcal{L}$. Starting from the initial point $(P_0, t_0) = (\bar{P}, 1)$, and parameter $\mu = \mu_0$,

1. Let $k = 0$.
2. Solve the unconstraint optimization problem
$$\min \Phi(P_k, t_k, \mu_k)$$
where $\Phi(P_k, t_k, \mu_k) := \mu_k t + b(P_k, t_k)$.
3. Let $\mu_{k+1} = (1 + \frac{1}{6K})\mu_k$.
4. Go to 1.

Of course, Step 2 cannot be solved exactly; a lot of research has been devoted to come up with a stopping criterion for this step which is sufficient for proving nice theoretical efficiency of the complete algorithm. One such criterion, rather interestingly, is to take '(one" Newton step, starting with a "nice" initial point, and then increase μ_k (the so-called short step method). Intuitively, as $k \to \infty$, $\mu_k \to \infty$, and the sequence of minimal Values of $\Phi(P_k, t_k, \mu_k)$ will approach $t_{\inf}$.

The resulting complexity bound below is the upshot of the interior point approach.

Theorem 2.1 ([13]) *For solving $\mathcal{L}$, starting with $(\bar{P}, 1)$ using*

$$O(K \log(K + \frac{1}{\epsilon})) \qquad (2.19)$$

iterations, the above barrier method computes (P^*, t^*), where $P^* \in \mathcal{F}_L$, and t^* is known to satisfy

$$\frac{t^* - t_{\inf}}{t_{\sup} - t_{\inf}} < \epsilon \qquad (2.20)$$

i.e., after $O(K\log(K+\frac{1}{\epsilon}))$, an ϵ-optimal point is found by the barrier method.

Theorem 2.1 has few implications, one of which is the following: If $t_{\inf} < \alpha < t_{\sup}$, after

$$O(K \log(K + \frac{t_{\sup} - t_{\inf}}{\min\{t_{\sup} - \alpha, \alpha - t_{\inf}\}})) \qquad (2.21)$$

iterations, for which the last pair is (P, t), is guaranteed to satisfy $P \in \mathcal{F}_L$ and $t = \alpha$.

Consider solving $\mathcal{L}$ by the interior point method described above. To check the stability of Σ_1 it is necessary and sufficient to stop the algorithm after the i-th iteration, when $t_i = 0$. According to (2) this is guaranteed after

$$\tau_{\Sigma_1} = O(K\log(K + \frac{t_{\sup} - t_{\inf}}{\min\{t_{\sup}, -t_{\inf}\}})) \qquad (2.22)$$

i.e., τ_{Σ_1} is the termination time of the barrier method for checking the stability of Σ_1. In view of the results reported in [13], wc proceed to show that,

> the product of τ_{Σ_1} and a particular robustness measure is a constant which depends only on the dimension of Σ_1, n.

Let $\bar{P} = \frac{1}{2}I$ and start the interior point method described earlier from $(\bar{P}, 1)$. Thus $t_{\sup} \geq 1$ and trivially $t_{\sup} < 2$. Note that $t_{\inf} \leq 0$, since if $t_{\inf} > 0$ and the pair $(t_{\inf}, P^*)$ is the solution to $\mathcal{L}$, then for $0 < \epsilon < 1$, $(\epsilon t_{\inf}, \epsilon P^*)$ is also a solution to $\mathcal{L}$, which is a contradiction.

Referring to (2.22), we observe that τ_{Σ_1} is essentially a function of K which is itself a function of n only, and a combination of $t_{\sup}$ and $t_{\inf}$. As Renegar observed in [13], $t_{\sup}$ and $t_{\inf}$ convey information shout a particular condition number or the distance to ill-posedness.

[1] In fact due to the result of Nesterov and Nemirovskii [10] every open convex domain in R^n has such a functional associated with it.

Translated in terms of the concepts iu stability analysis, this parameter is in fact a "robustness" measure, as we proceed to show below.

Let $\alpha := \inf_\Delta \frac{\|\Delta\|}{\|A\|}$ such that $A + \Delta$ is not Hurwitz. Define " "

$$\delta_{\Sigma_1} := \frac{1}{\log(\sqrt{n} + \frac{1}{\alpha})} \qquad \textbf{(2.23)}$$

Since -1< $t_{\inf} \leq 0$, for small ϵ >0,

$$A'P + PA < (t_{\inf} - \epsilon)(A'\hat{P} + \hat{P}A + I) \qquad \textbf{(2.24)}$$

$$P > 0 \qquad \textbf{(2.25)}$$

$$\|P\| < 1 \qquad \textbf{(2.26)}$$

is inconsistent. 'J'bus,

$$\alpha\|A\| \leq |t_{\inf} - \epsilon|\|A + I\| \qquad \textbf{(2.27)}$$

$$\leq (|t_{\inf}| + \epsilon)(\|A\| + 1 \qquad (2.28)$$

$$\Rightarrow \frac{\alpha\|A\|}{\|A\| + 1} \leq |t_{\inf}| + \epsilon \qquad \textbf{(2.29)}$$

$$\Rightarrow t_{\inf} \leq \frac{-\|A\|}{\|A\| + 1}\alpha \qquad \textbf{(2.30)}$$

since $t_{\inf} \leq$ O. The first inequality above is the result of two propositions reported in [1 **3].**

Since O $\leq -t_{\inf} \leq t_{\sup} \leq 2$,

$$\tau_{\Sigma_1} = O(K \log(K + \frac{3}{-t_{\inf}}))$$

$$= O(K \log(K + \frac{1}{\alpha}))$$

Thereby wc have established the following theorem.

Theorem 2.2

$$\tau_{\Sigma_1} \delta_{\Sigma_1} = O(K) = O(\sqrt{n}) \qquad (2.31)$$

Theorem 2.2 constitutes a natural, but very interesting relationship between robustness properties of Σ_1 and the efficiency of ipms for determining whether Σ_1 is stable. More specifically, given that (2.31) holds, fixing n and using the ipm for the solution of the Lyapunov equation, certain information pertaining to the relative stability of the corresponding system is somehow revealed! This observation has consequences which go far beyond the stability analysis of Σ_1.

2.3

Complexity analysis in terms of " condition measures" has interesting implications for problems considered in system and control theory. This is iu part due to the fact that the interior point methods (ipms) can iu principle be applied to all convex optimization problems. In this section we shall provide another example which reinforces our belief that the conceptual framework developed in the previous subsection has far reaching consequences, this time in the context of checking the positive realness of a transfer matrix. In this avenue, wc first state the (generalized) Positive Real (GPR) Lemma, as well as its consequences iu studying the absolute stability problem. Then two robustness measures for the positive real systems arc discussed, one of which corresponds to the notion of gain margin. Having discussed the GPR Lemma and its applications in stability analysis, au exact analogue of the interior point method and its computational efficiency presented of Section 2.2 arc discussed for the LMI arising from the GPR Lemma. in particular, it is shown that the product of the running time of the ipm and a certain robustness measure for GPR systems, is a constant which depends only on the dimension of the underlying system.

Consider the linear time invariant system Σ_2:

$$\Sigma_2 : \quad \dot{x} = A\,x + B\,u \qquad \textbf{(2.3'2)}$$

$$y = C\,x + D\,u \qquad \textbf{(2.33)}$$

such that the quadruple (A, B, C, D) is the minimal realization of the transfer matrix

$$H(s) = C(SI - A)^{-1}B + 1)$$

in which case wc write $H \sim (A, B, C, D)$. Wc shall assume that the pairs (A, B) and (A, C) arc respectively, controllable and observable. The matrix A is also assumed to be Hurwitz. For further reference let $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{m\times n}$ and $D \in R^{m\times n}$ and without 10ss of generality assume that $m < n$. Given an initial condition x_0 aud a control function η which maps y to u, the equations (2.32)- (2.33) define a trajectory for the feedback system Σ_2.

Given that $H \sim (A, B, C, D)$ and assuming that A is Hurwitz, the transfer matrix 11 is called (generalized) strongly positive real (GSPR) if there exists c > 0 such that

$$H(j\omega) + H^*(j\omega) > \epsilon I \; \forall \omega$$

where $H^*(j\omega)$ denotes the conjugate transpose of the transfer matrix $H(j\omega)$. The (generalized) Positive Real (GPR) Lemma states that $H(s)$ is GSPR and stable if and only if the following system of linear matrix inequalities is feasible [1], [14],

$$\mathrm{herm}\{\begin{pmatrix} P & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\} < 0 \quad (2.34)$$

$$1' > 0 \qquad \textbf{(2.35)}$$

Let us define two robustness measures for a GSPR system. Denote by $E := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ " Now let,

$$\alpha := \inf_\Delta \|\Delta\|/\|E\| \qquad (2.36)$$

such that there (foes *not* exist a matrix P that satisfies the following set of linear matrix inequalities,

$$\mathrm{herm}\{\begin{pmatrix} P & 0 \\ 0 & -I \end{pmatrix}(E+\Delta)\} < 0 \quad (2.37)$$

$$P > 0 \quad (2.38)$$

and let

$$\delta^1_{\Sigma_2} := 1/\log(\sqrt{n}+m+\frac{1}{\alpha}) \quad (2.39)$$

The quantity $\delta^1_{\Sigma_2}$ is a measure of the relative perturbation that a stable GSPR system Σ_2 can tolerate, and remain stable and GSPR. The perturbation A, can for example be the result of the uncertainty in the modeling of the plant, or due to tile finite accuracy of the computer arithmetic (which for example, is used to check the GSPR property of the system).

Our next robustness measure is defined to be

$$\delta^2_{\Sigma_2} := 1/\log(\sqrt{n+m}+\frac{1}{\beta}) \quad (2.40)$$

where β is defined by the following optimization problem:

$$\beta := \inf \lambda \quad (2.41)$$

$$\text{s.t. } \mathrm{herm}\{\begin{pmatrix} P & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} A & B \\ C & \lambda I + D \end{pmatrix}\} < 0 \quad (2.42)$$

$$P > 0 \quad (2.43)$$

$$\|P\| < 1 \quad (2.44)$$

$$\lambda \geq 0 \quad (2.45)$$

Let us provide a motivation for introducing $\delta^2_{\Sigma_2}$. Consider the feedback system consisting of Σ_2 in the forward path, and a nonlinear time invariant control function η in the feedback path, i.e., $u = -\eta(y)$. Assume furthermore that η belongs to the sector $[0, 1/k]$, for some real positive number k, i.e., $0 \leq y'\eta(y) \leq k\|\eta(y)\|^2$, for all $y \in R^m$. The quantity $\delta^2_{\Sigma_2}$ is essentially a measure of the maximal sector which η can belong to, such that the close loop system is guaranteed to be absolutely stable via the GPR Lemma.

The two *robustness measures* for a GPR systems just introduced are related in an interesting way to the *computational efficiency* of the barrier method (when applied to solve the system of LMIs malting from the GPR Lemma). The relationship is of the following form:

> the products of the running time of the barrier method (with suitably chosen initial points) and the robustness measures $\delta^1_{\Sigma_2}$ and $\delta^2_{\Sigma_2}$ arc constants which depend only on the *dimension of* Σ_2.

Theorem 2.3 *Given the system Σ_2, there is an algorithm $\mathcal{A}$ such that for the robustness measures $\delta^1_{\Sigma_2}$ and $\delta^2_{\Sigma_2}$,*

$$\delta^1_{\Sigma_2}\tau_{\mathcal{A}(\Sigma_2)} = c_1$$

and

$$\delta^2_{\Sigma_2}\tau_{\mathcal{A}(\Sigma_2)} = c_2$$

for some constants c_1 and c_2 which depend 071131071 $n+m$.

Proof: We consider the problem of verifying whether a transfer matrix is GSPR using the barrier method discussed in the previous section. For this purpose, we apply the method to solve the system of matrix inequalities (2.34)-(2.35).

In complete analogy with the application of ipms for solving the Lyapunov equation, in order to find a feasible point of the inequalities (2.34)-(2,35), we apply the barrier method to solve,

$$t = 0$$

$$\mathrm{herm}\{\begin{pmatrix} P & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\}$$

$$< t\,(\mathrm{herm}\{\begin{pmatrix} \frac{1}{2}I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\} + \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix})$$

$$P > 0$$

$$\|P\| < 1$$

$$-1 < t < 2$$

Note that $(\sim I, 1)$ is a feasible point for the set defined by the last four inequalities above.

By replacing A with E in the analysis presented in Section 2.2 and repeating the exact sequence of arguments, we conclude that the number of iterations needed to obtain a feasible point of the set defined by the above equality and inequalities, and consequently to check the GSPR property of Σ_2 is,

$$O(K\log(2+K+1/\alpha))$$

where $K = \sqrt{n+m}$. In view of the definition of $\delta^1_{\Sigma_2}$, it follows that

$$\tau_{\mathrm{ipm}(\Sigma_2)}\delta^1_{\Sigma_2} = c_1$$

where c_1 is a constant which depends only on the dimension of the system (through the variable K).

Interestingly, by an appropriate choice of the initial point for the barrier method, its running time can be made to be inversely proportional to the robustness measure $\delta^2_{\Sigma_2}$ [8]. □

3 Concluding Remarks

The main purpose of the paper is to point out a very close relationship between stability analysis of dynamical systems on one hand, and the theoretical studies on the efficiency of certain numerical algorithms. in particular, we have demonstrated that for very important stability problems, the efficiency of the interior point methods, can convey certain information about the relative stability of the corresponding systems. These results arc due to the existence of a self-co~lc.orclant barrier for the cone of positive semi-definite matrices, with a self-collcordant parameter which depends only on the dimension of the space for which the problem is formulated in. This phenomena can in principle be used to give an *algorithmic definition* of the relative stability of a dynamical system.

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